Discharge Coefficient of a Chemical Laser Nozzle

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Introduction

THE throat half-width h^* of a two-dimensional chemical laser nozzle can be as small as 0.005 cm. At moderate or low plenum pressures the Reynolds number is small, and the thickness of the laminar boundary layer at the throat is substantial. For the oxidizer nozzles, in particular, a knowledge of the throat displacement thickness $(\delta^*)^*$ is important for several reasons. Determination of plenum conditions, especially the fraction of undissociated fluorine, depends on the discharge coefficient C_D, here defined for a two-dimensional nozzle as $C_D = 1-(\delta^*)^*/h^*$. A second reason for establishing $(\delta^*)^*$ is to replace the zero-value assumption often used to initiate the boundary-layer calculation for the divergent part of the nozzle with a more realistic value for $(\delta^*)^*$. In contrast to stagnation-point flow, the pressure gradient parameter β (defined later) does not have a unique value at the throat, and consequently it is also important to establish β^* for this calculation. (An asterisk denotes throat conditions.)

Cold flow tests, where there is little or no wall heat transfer, have shown the dependence of C_D on Reynolds number Re.[1,2] As expected these tests show $(\delta^*)^* \propto (Re)^{-1/2}$, and C_D is substantially below unity at Reynolds numbers of importance for chemical lasers. For the oxidizer nozzles, however, the wall is water cooled while the core flow is hot (say, 1200 K), and wall heat transfer substantially changes $(\delta^*)^*$ from its cold flow value.

We first theoretically explore, in the simplest manner possible, the determination of $(\delta^*)^*$ and β^*. Based on the results of this analysis, a new test procedure is suggested that would establish "effective" values for these parameters.

Analysis

The parameter β^* is generally large compared to unity.[3] Coles,[4] therefore, used matched asymptotic expansions, with $\beta^{-1/2}$ as the small parameter, to delineate the structure of a similar, compressible, laminar boundary layer. He further assumed viscosity proportional to temperature, constant wall temperature, Prandtl number of unity, zero velocity at the wall, and the flow of a perfect gas in a two-dimensional nozzle. The lowest-order solution was obtained for the inner and outer layers for arbitrary values of the wall to stagnation temperature ratio $t_w [=(T_w/T_0)]$. The displacement thickness was obtained, however, only for the special case of an adiabatic wall, i.e., $t_w = 1$. Cole's solution for the leading term of the outer expansion is approximate; Back and Witte,[3] however, show good agreement between the approximate solution and numerically obtained one for both the wall heat transfer and shear. Beckwith and Cohen[5] and Dewey and Gross[6] also have examined the large β limit, but did not evaluate the displacement thickness. Back[7] has presented numerically obtained results for a wide range of boundary-layer parameters, including large β. Although these results are not readily interpreted directly in terms of throat conditions, the observed trends are in accord with the results presented here.

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For simplicity and conciseness, the assumptions of Coles[4] are retained. The boundary-layer transformations and notation, however, are the same as used by Ref. 6, which have become standard. The boundary-layer equations are then given by

$$f''' + ff'' = \beta[(f')^2 - (1-t_w)\theta - t_w] \tag{1a}$$

$$\theta'' + f\theta' = 0 \tag{1b}$$

$$f(0) = f'(0) = \theta(0) = 0 \tag{2a}$$

$$f'(\infty) = \theta(\infty) = 1 \tag{2b}$$

where

$$f' = \frac{u}{u_e}, \quad \theta = \frac{H - H_w}{H_e - H_w}$$

$$\xi = \int_{x_i}^{x} \rho_w \mu_w u_e dx, \quad \eta = \frac{u_e}{(2\xi)^{1/2}} \int_0^y \rho dy \tag{3}$$

$$\beta = \frac{2\xi}{u_e} \frac{du_e}{d\xi} \frac{T_0}{T_e} \tag{4}$$

In the aforementioned, x is distance along the nozzle, x_i is the origin of the boundary layer, and H is the total enthalpy. All symbols are defined in Ref. 6, and have their usual boundary-layer meaning. The displacement thickness can be shown to be given by[6]

$$\frac{(\delta^*)^*}{h^*} = \left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/2(\gamma-1)} \frac{(2\xi^*)^{1/2}}{(\rho a)_0 h^*} I^*(\beta^*, t_w, \gamma) \tag{5}$$

where a is the speed of sound, and a zero subscript denotes freestream stagnation conditions. The integral I^* is given by

$$I^* = \left(\frac{\gamma+1}{2}\right) \int_0^\infty [(f')^2 - (1-t_w)\theta - t_w] d\eta - \int_0^\infty f'(1-f') d\eta \tag{6}$$

and thus depends only on $f'(\eta)$ and $\theta(\eta)$.

A procedure is used for the evaluation of β^* similar to that in Ref. 3 for an axisymmetric nozzle. For this a simple nozzle geometry is used

$$\frac{A}{A^*} = 1 + \left(\frac{z}{z^*}\right)^2 = \frac{h}{h^*} \tag{7}$$

where z is axial distance, Fig. 1, and A is the nozzle's cross-sectional area. The characteristic length z^* is related to the radius of curvature at the throat r^* and h^* by $z^* = (2h^*r^*)^{1/2}$. The x and z coordinates are related by the transformation

$$dx = \left[1 + \left(\frac{z}{r^*}\right)^2\right]^{1/2} dz \tag{8}$$

with both x and z equal to zero at the throat.

To evaluate β^* it is necessary to determine ξ^*. Toward this end, introduce Eq. (8) and Cole's assumptions

$$\mu_w = \frac{\mu_0}{T_0} T_w, \quad \rho_w = \rho_e \frac{T_0}{T_w} \left(1 + \frac{\gamma-1}{2} M_e^2\right)^{-1}$$

into Eq. (3) to obtain

$$\xi^* = \frac{\mu_0}{A^*} \int_{z_i}^{0} (\rho u)_e A \frac{[1 + (z/r^*)^2]^{1/2}}{(A/A^*)[1 + (\gamma-1)M_e^2/2]} dz \tag{9a}$$

The nozzle mass flow rate is $(\rho u)_e (A - 2\delta^*)$, and with the assumption of $(\delta^*/h) \leq 0.1$ in the convergent part of the noz-

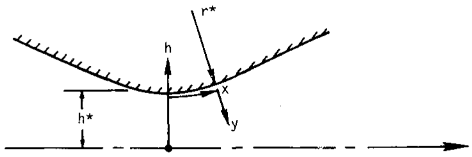

Fig. 1 Schematic of nozzle geometry.

zle, is approximately $(\rho u)_e A \simeq$ constant. Furthermore, the quantity, $1+[(\gamma-1)/2]M_e^2$, varies only slightly in the subsonic part of the nozzle, and can be approximated by its average value of $(\gamma+3)/4$. Hence, ξ^* is approximated by

$$\xi^* \simeq \frac{4}{(\gamma+3)}\left(\frac{\gamma+1}{2}\right)^{-(\gamma+1)/2(\gamma-1)}(\rho a \mu)_0 z_i J \tag{9b}$$

and with the use of Eq. (7), J is

$$J=\int_{z_i}^{0}\frac{[1+(z/r^*)^2]^{1/2}}{1+(z/z^*)^2}\frac{dz}{z_i}$$

This integral is related to the elliptic integral of the third kind and can be shown to equal[8]

$$J=\frac{a}{b^2}\left\{ln[a+(1+a^2)^{1/2}] + \frac{(a^2-b^2)^{1/2}}{a} ln\left[\frac{(1+b^2)^{1/2}}{(1+a^2)^{1/2}+(a^2-b^2)^{1/2}}\right]\right\}, a\geq b$$

$$J=\frac{a}{b^2}\left\{ln[a+(1+a^2)^{1/2}] + \frac{(b^2-a^2)}{a}\tan^{-1}\left(\frac{b^2-a^2}{1+a^2}\right)^{1/2}\right\}, a\leq b$$

where $a=-(z_i/r^*)$, $b=-(z_i/z^*)$, and z_i is negative.

The pressure gradient at the throat is given by[9]

$$\left(\frac{1}{p}\frac{dp}{dz}\right)^* = -\gamma\left[\frac{(d^2A/dz^2)^*}{(\gamma+1)A^*}\right]^{1/2} = -\frac{\gamma}{[(\gamma+1)h^*r^*]^{1/2}}$$

Thus, $(du_e)^*$ and $(d\xi)^*$ are given by

$$(du_e)^* = -\frac{(dp_e)^*}{(\rho u)_e^*} = \frac{2}{\gamma+1}a_0\frac{dz}{z^*}$$

$$(d\xi)^* = (\rho_w \mu_w u_e)^* dx = \left(\frac{\gamma+1}{2}\right)^{-(3\gamma-1)/2(\gamma-1)}(\rho a \mu)_0 dx$$

Since $dz=dx$ at the throat, the velocity gradient is

$$\left(\frac{du_e}{d\xi}\right)^* = \left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/2(\gamma-1)}\frac{1}{(\rho\mu)_0 z^*} \tag{10}$$

Inserting Eqs. (9b) and (10) into Eq. (4) and simplifying, produces

$$\beta^* = \frac{[2(\gamma+1)]^{3/2}}{\gamma+3}bJ(a,b) \tag{11a}$$

and we have the important result that β^* depends only on γ, throat geometry, and an inlet length.

In the special case when $r^*=2h^*$, we have $a=b$ and Eq. (11a) simplifies to

$$\beta^* = \frac{[2(\gamma+1)]^{3/2}}{\gamma+3} ln[a+(1+a^2)^{1/2}] \tag{11b}$$

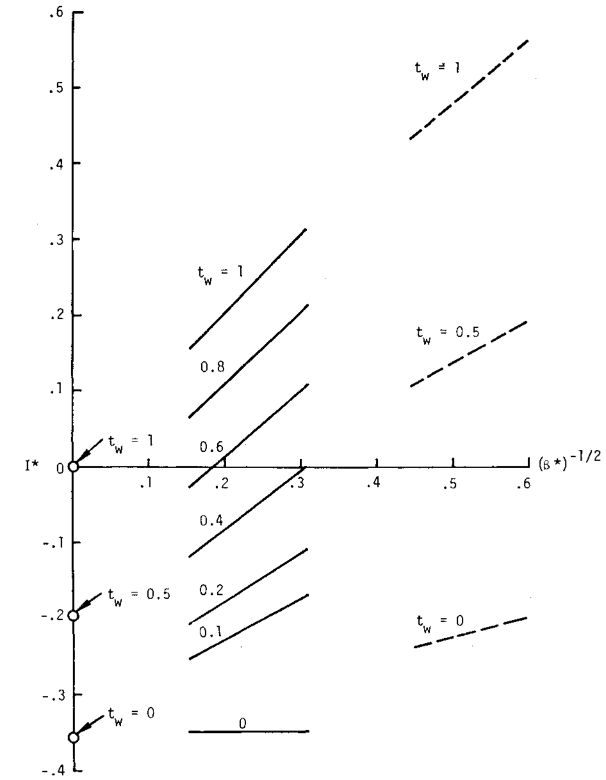

Fig. 2 I^* vs $(\beta^*)^{-1/2}$ for $\gamma=1.4$. The values at $\beta^*=\infty$ and the dashed curves are from Ref. 6.

Thus, β^* becomes logarithmically infinite as $a=-(z_i/r^*)\to\infty$. It is worth noting that the integral in Eq. (9a) is finite when integrated from $-\infty$ to zero if the incorrect assumption $dx \simeq dz$ is used. As a consequence, Eq. (11a) would not contain either a or b; β^* would depend only on γ, and the large β^* limit would be inappropriate.

The displacement thickness is obtained by combining Eqs. (5), (9b), and (11a)

$$\frac{(\delta^*)^*}{h^*} = \left(\frac{\gamma+1}{2}\right)^{(2-\gamma)/2(\gamma-1)}\left(\frac{2r^*}{h^*}\right)^{1/4}\left(\frac{\beta^*}{Re_0}\right)^{1/2} I^* \tag{12}$$

where $Re_0=(\rho a)_0 h^*/\mu_0$. All that remains is to evaluate I^*, Eq. (6). For this, a uniformly valid, additive composite solution is used based on Coles,[4] lowest-order inner and outer expansions. This solution, denoted by a c subscript, is given by

$$f_c' = [(1-t_w)\text{erf}(C\eta/2^{1/2})+t_w]^{1/2} - 3t_w^{1/2}\text{sech}^2\{t_w^{1/4}(\beta/2)^{1/2}\eta+\tanh^{-1}[(2/3)^{1/2}]\} \tag{13a}$$

$$\theta_c = \text{erf}(C\eta/2^{1/2}) \tag{13b}$$

$$C=\left(\frac{2}{3}\frac{t_w^{3/2}-1}{t_w-1}\right)^{1/2}$$

A number of features characterize this solution that are important in the latter discussion:

1) The lowest-order outer expansion is equivalent to $(\rho u^2)_{out}=(\rho u^2)_e$, or $M_{out}=1$ at the throat. In the special case when $t_w=1$, $C=1$, $(\rho u)_{out}=(\rho u)_e$, and the outer expansion provides no contribution to $(\delta^*)^*$. When $t_w=1$, only the inner expansion contributes and produces a positive value for $(\delta^*)^*/h^*$ of order unity.[4] (The integrals in Eq. (6) are easily evaluated exactly when $t_w=1$.)

2) When $t_w=0$, the outer expansion satisfies wall conditions (2a) and is thus uniformly valid. In this case only the

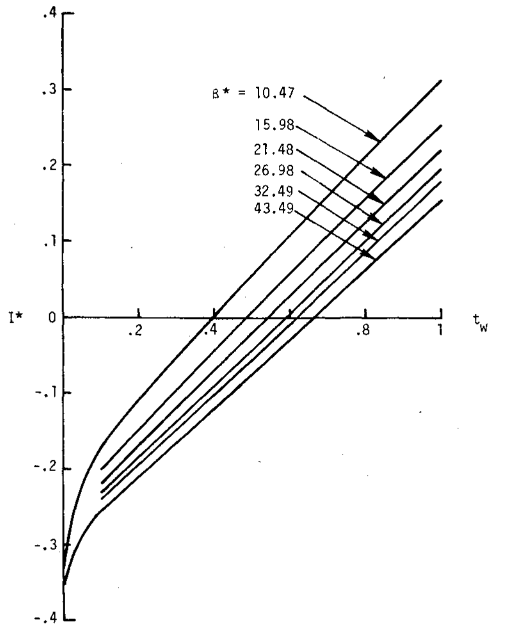

Fig. 3 I^* **vs** t_w **for** $\gamma = 1.4$. **The** β^* **values are given by Eq. (11b), with** $a = 40,400, \ldots$.

outer expansion contributes to $(\delta^*)^*$, and produces a negative value for $(\delta^*)^*$.

Results

In contrast to the difficult problem of numerically solving Eqs. (1) for large β,[6] it is quite easy to integrate (via Simpson's rule) Eqs. (13) to obtain I^*. Figure 2 shows results for $\gamma = 1.4$ and results from Ref. 6 at low β^* (dashed curves) and at $\beta^* = \infty$. Agreement is poor for $t_w = 0$, were only the outer layer contributes to I^*. Inclusion of the next term in the outer expansion would improve this comparison. This figure clearly shows I^* proportional to $(\beta^*)^{-1/2}$ when $t_w = 1$. Since $(\delta^*)^*$ is proportional to $(\beta^{1/2} I)^*$, Eq. (12), then $(\delta^*)^*$ itself is independent of β^*. Thus, a cold flow measurement of $(\delta^*)^*$, when $t_w = 1$, cannot be used to determine β^*.

When $0 < t_w < 1$, I^* consists of a term dependent on γ and t_w and a $(\beta^*)^{-1/2}$ term, with the $(\beta^*)^{-1/2}$ term vanishing when $t_w = 0$. As is evident from Eq. (12), $(\delta^*)^*$ is positive when I^* is positive. As expected, $(\delta^*)^*$ becomes negative, and C_D exceeds unity, for a sufficiently large β^* when $0 < t_w < 1$. A typical value for β^* is 20,[3] and a hot-flow value for $(\delta^*)^*$ (say at $t_w = 0.2$) differs from its cold-flow value by the large multiplicative factor of -1.39, and, consequently, the hot-flow discharge coefficient exceeds unity.

Figure 3 shows the solid curves for I^* replotted against t_w with β^* as the parameter. This figure suggests the following experimental procedure for determining "effective" values for β^* and $(\delta^*)^*$. Extend the range of t_w values by using a heater to preheat the He or N_2 normally used in cold-flow C_D tests. By testing over a range of plenum temperatures, with the wall temperature at the throat more or less fixed by the water cooling, the dependence of C_D on t_w is established. With γ, h^*, and r^* known, and Re_0 readily computed for each flow condition, Eq. (12) then provides the dependence of $(\beta^{1/2} I)^*$ on t_w. By comparing these experimentally derived values with corresponding theoretical ones, an effective value for β^* is established. (This procedure produces only an "effective" value, since the theory, for example, may assume unity Prandtl number, or an estimated value for r^*.)

As shown by Fig. 3, I^* is nearly linear with t_w down to about 0.1. Thus, the foregoing C_D vs t_w data, when plotted as I^* vs t_w, can be extrapolated to lower t_w values (down to 0.1) to yield a C_D applicable to an actual laser flow, where t_w is typically 0.2 to 0.3.

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Incipient Separation of Leeward Flow Past a Lifting Plate in Viscous Hypersonic Flow

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Nomenclature

C = Chapman-Rubesin viscosity constant
c = chord length
M = Mach number
Re = Reynolds number based on freestream conditions
α = angle of incidence
α^* = critical angle of incidence, Eq. (1)
χ = hypersonic viscous-inviscid interaction parameter, $(M_\infty^3 \sqrt{C}/\sqrt{Re_c})$

Subscripts

∞ = conditions in the freestream
c = based on chord length, corner position or step face, in the interaction model
exp = experimental value

Introduction

It is well known that when a thin flat plate of finite chord is set at incidence to an oncoming supersonic/hypersonic stream, depending on the angle of incidence, flow Mach num-

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